Indian Statistical Institute
Semestral Exam
Algebra-I
26-11-2010

Time : 3 hours | Max. Marks : 60

Answer question **1** and **any 5** from the rest. All questions carry equal marks.

(1) State true or false. Justify your answers briefly.
(a) There are 48 elements of order 7 in a simple group of order 168.
(b) Any group of order pq where p, q are distinct primes, is cyclic.
(c) The additive group $\mathbb{Z}$ is isomorphic to $H \times K$, for some non-trivial subgroups H, K of $\mathbb{Z}$.
(d) D_6, the dihedral group of order 6, is isomorphic to S_3.
(e) A group of order p^n, p a prime and $n \geq 1$, has non-trivial center.

(2) Show that two elements in S_n are conjugates in S_n if and only if they have the same cycle type. Hence compute the order of the centraliser $C_{S_n}(\sigma)$ of an m-cycle σ in S_n.

(3) Let G be a group of order p^n where p is a prime and let G act on a finite set X. If $X_0 = \{x \in X / \ gx = x \ \forall g \in G\}$ then show that $|X| \equiv |X_0| \mod p$. Hence show that if p does not divide $|X|$ then there exists an element in X which is fixed by all elements of G.

(4) Let $G' =< x^{-1}y^{-1}xy \ / \ x, y \in G >$ be the commutator subgroup of G. Show that
(a) G' is normal in G,
(b) A subgroup H contains G' if and only if H is normal in G and G/H is abelian.

(5) Let G be a group of order 255. Show that
(a) G is not simple,
(b) If H is a 17-Sylow subgroup of G then $H \subseteq Z(G)$.

(6) Let G be a finite abelian group. Show that G is the internal direct product of its Sylow subgroups.

(7) Using simplicity of A_n for $n \geq 5$, show that A_n is the only subgroup of index 2 in S_n, for all $n \geq 3$.